Fifteenth-century Italian symbolic algebraic calculation with four and five unknowns

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A Dario e Francesca,
cari amici

ABSTRACT

The present article continues an earlier analysis of occurrences of two algebraic unknowns in the writings of Fibonacci, Antonio de' Mazzinghi, an anonymous Florentine abbacus writer from around 1400, Benedetto da Firenze and another anonymous Florentine writing some five years before Benedetto, and Luca Pacioli. Here I investigate how in 1463 Benedetto explores the use of four or five algebraic unknowns in symbolic calculations, describing it afterwards in rhetorical algebra; in this way he thus provides a complete parallel to what was so far only known (but rarely noticed) from Michael Stifel's *Arithmetica integra* (1544) and Johannes Buteo's *Logistica* (1559). It also discusses why Benedetto may have seen his innovation as a merely marginal improvement compared to techniques known from Fibonacci's *Liber abbaci*, therefore failing to make explicit that he has created something new.

Opening remarks

In a recent publication [Høyrup 2019],[1] I have discussed little-noticed appearances of the use of two algebraic unknowns in Latin Europe (as a matter of fact, Tuscany and Venice only) from Leonardo Fibonacci until Luca Pacioli.[2] Some of them come from Benedetto da Firenze's *Trattato de praticha d'arismetrica* (autograph Siena, Biblioteca degl'Intronati L.IV.21) from 1463, others from his undated *Tractato d'abbacho* [Arrighi 1974].[3]

The *Praticha* encompasses 496 densely folio sheets – counted by letters around 2.25 as much as Fibonacci's *Liber abbaci* and 1.2 times as much as Luca Pacioli's *Summa*. It was prepared – thus the introductory passage on fol. 1^r – as a gift to "a

Gaṇita Bhāratī Vol. 42, No. 1-2 (2020) pages 55-86
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DOI-https://doi.org/10.32381/GB.2020.42.1-2.3

dear friend" in 1463. But this friend can be seen to be one with whom Benedetto could only communicate indirectly, so the relation was clearly one of patronage. Fortunately the same first page carries the coat of arms of the Marsuppini family, showing us that Benedetto's protector belonged to the absolute top of the Florentine patriciate.

In the article in question I discussed only one of the problems from the *Praticha d'arismetricha* that make use of two unknowns (though showing a marginal calculation belonging with another problem and also mentioning three similar problems from the *Tractato d'abbacho*). There are many more. Mostly, the unknowns appear (in both treatises) as *borsa* and *quantità* or *chavallo* and *quantità*. The former couple is used when several men having money find a purse (*borsa*), the latter when they want to buy a horse (*chavallo*). A few examples in the *Praticha* (fols 222^v, 249^v, 254^v, 255^r) call the unknowns *chosa*, "thing", and *quantità* instead of *borsa* and *quantità* or *chavallo* and *quantità*. Inclusion of these in my discussion would have made it exorbitantly long but not have changed the global picture much.

The problems which I discussed are accompanied by marginal calculations making use of *b* (when a *borsa* is involved) and *q* (for the *quantità*). They can be seen to have been made before the calculation was described in words in the regular text – that is, it is actually the regular text which accompanies and explains the marginal calculations, not *vice versa*. Being made *before* and thus independently of the textual explanation, these marginal calculations can be said to represent *symbolic* algebra, as defined by Nesselmann [1842: 302].[4] In symbolic algebra, as he explains, we may

> develop an algebraic argument from beginning to end in completely understandable way without needing any written word, and indeed, at least in simpler developments, only occasionally insert a conjunction between the formulas.

Four men and a purse – first use of a new technique

In the present article I shall discuss something that goes further. As a first step I present the use of no less than five algebraic unknowns in a symbol-based resolution of a problem from Benedetto's *Praticha* about four men finding a purse. I give a translation of the text of the problem, accompanied by the first appurtenant manuscript page (where the symbolic calculations are found) redrawn for clarity; next comes a detailed mathematical paraphrase, and then a comparison with Fibonacci's treatment of the same problem in the *Liber abbaci* that will allow us to see why Benedetto might look at his innovation as only marginal. The second part of the article looks at how Benedetto develops the idea until it seems ripe, and then leaves things there. An appendix contains a transcription of the original text of the problems that have been discussed.

The first problem runs like this:[5]

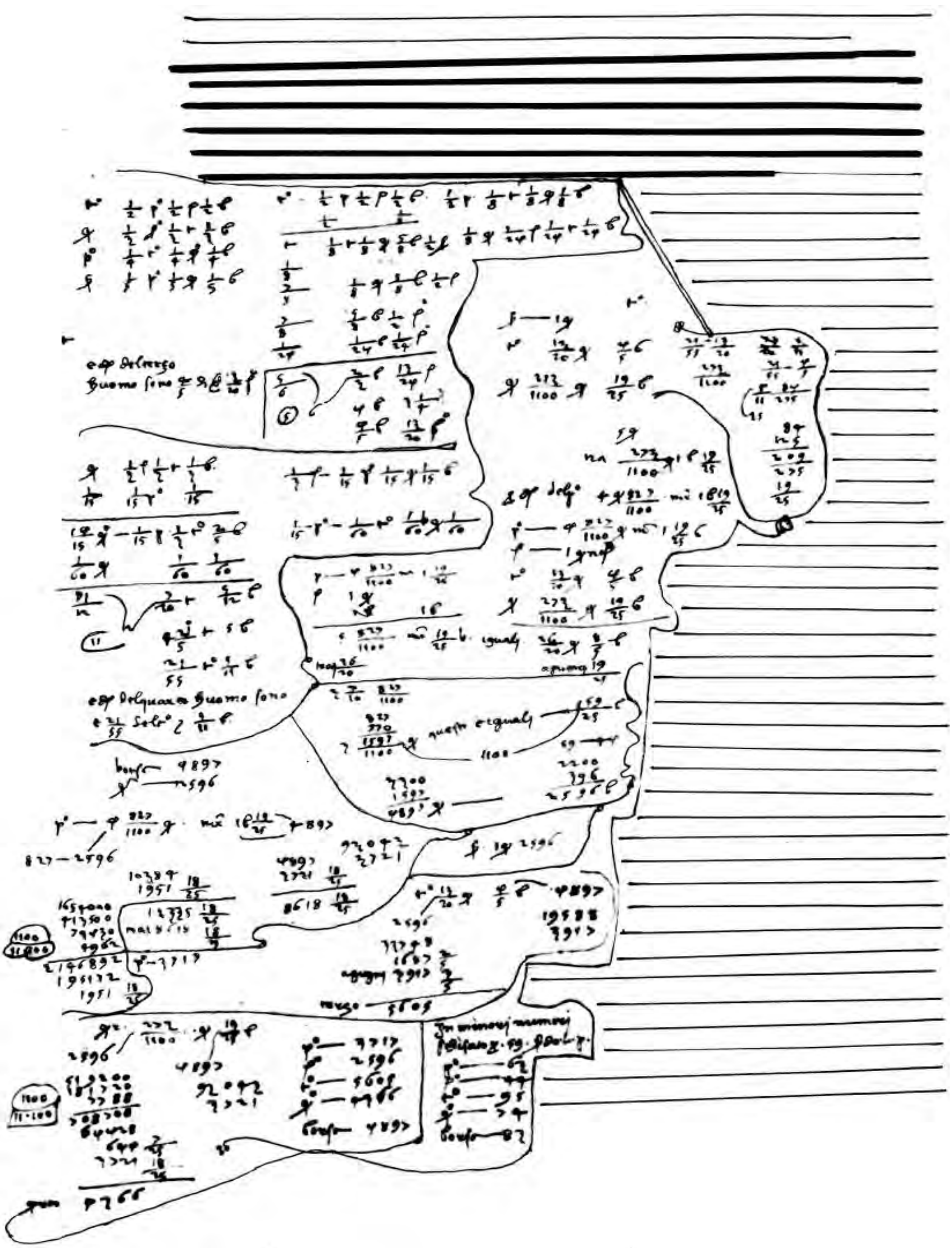
> (fol. 270v)Four have *denari*,[6] and walking on a road they found a purse with *denari*. The first and second say to the third, if you give us the purse we shall have 2 times as much as you. The second and third men say to the fourth, if we had the *denari* of the purse we should have 3 times as much as you. The third and fourth say to the first, if we had the *denari* of the purse we should have 4 times as much as you. The fourth and the first say to the second, if you give us the *denari* of the purse we shall have 5 times as much as you. It is asked how much each one had, and how many *denari* there were in the purse. We shall do in this way, you shall say, the first and the second with the *denari* of the purse say to have 2 times as much as the third man. Whence the third man by himself had the[7] $^1/_2$ of that which the first and the second and the purse have. And mark (*segnia*[8]) this. And then you shall say, the second and the third man with the purse have 3 times as much as the fourth, so the fourth man had the $^1/_3$ of that which the first and second have, and of the purse. And mark even this. And then you shall say, the third and the fourth man with the purse had 4 times as much as the first, and therefore the first man by himself had the $^1/_4$ of that which the third and the fourth man had, and of the purse. And mark this. And then you shall say, the fourth and first have with the purse 5 times as much as the second, so that the second will have the $^1/_5$ of that which the first and fourth man have, and of the purse. And this is marked. And you shall bring the *denari* of the third to a comparison.[9] And you shall say, the *denari* of the third man is as much as the $^1/_2$ of the *denari* of the first and the second and of the purse. From where it is to be known, how much are the $^1/_2$ of the *denari* of the first, which we have ¿brought together,? that the *denari* of the first are the $^1/_4$ of the *denari* of the third and fourth man and of the purse, and let the $^1/_2$ of the *denari* of the first be $^1/_8$ of the *denari* of the third and fourth and of the purse. Therefore you shall say that the *denari* of the third should be as much as the $^1/_2$ of the *denari* of the second and of the purse and as much as $^1/_8$ of the *denari* of the third and fourth and of the purse. Therefore you shall take away $^1/_8$ of the *denari* of the third and join $^1/_8$ of purse to $^1/_2$ purse. And we shall have that $^7/_8$ of the *denari* of the third are $^1/_8$ of the *denari* of the fourth and $^1/_2$ of the *denari* of the second and $^5/_8$ of purse. And then (fol. 271r) it is to be seen how much $^1/_8$ of the *denari* of the fourth are, where we say that all the *denari* of the fourth man are $^1/_3$ of the second

and the third man and of the purse. Therefore, $^1/_8$ of the *denari* of the third man will be $^1/_{24}$ of the *denari* of the second and of the third and of the purse. Therefore, you shall take away from $^7/_8$ of the *denari* of the third $^1/_{24}$ of the denari of the third man and above $^5/_8$ of purse you shall put $^1/_{24}$ of purse, and above $^1/_2$ of the second you shall put $^1/_{24}$ of the *denari* of the second, and we shall have that $^5/_6$ of the *denari* of the third man are as much as the $^2/_3$ of the purse and $^{13}/_{24}$ of the *denari* of the second. And you shall say, if $^5/_6$ of the *denari* of the third man are as much as the $^2/_3$ of the purse and $^{13}/_{24}$ of the second, how much will all the *denari* of the third be? Where you shall divide $^2/_3$ of the *denari* of the purse and $^{13}/_{24}$ of the *denari* of the second in $^5/_6$, from which comes $^4/_5$ of the *denari* of the purse and $^{13}/_{20}$ of the *denari* of the second. This is taken note of (*notato*[10]). And in the same way you shall make the *denari* of the fourth. It is true that it could be done without that, but since it is in the castelet[11] this order shall be pursued. You shall say, the fourth has the $^1/_3$ of the *denari* of the second and the third and of the purse. Where you shall get the $^1/_3$ of the *denari* of the second without the others, and you shall say, the second has as much as the $^1/_5$ of the first and of the fourth and of the purse. Wherefore the $^1/_3$ of the *denari* of the second will be $^1/_{15}$ of the *denari* of the first and of the fourth and of the purse. Where from the *denari* of the first you shall take away $^1/_{15}$ and you shall join to $^1/_3$ of purse $^1/_{15}$ of purse, and we shall have that $^{14}/_{15}$ of the fourth are as much as $^1/_{15}$ of the first and $^1/_3$ of the third and $^2/_5$ of purse. Then $^1/_{15}$ of the first will be $^1/_{60}$ of the third and $^1/_{60}$ of the fourth and of the purse, where from $^{14}/_{15}$ of the *denari* of the fourth take away $^1/_{60}$, and above $^1/_3$ of the *denari* of the third man join $^1/_{60}$ of the *denari* of the third man, and above the $^2/_5$ of the *denari* of the purse put $^1/_{60}$ of purse, and you shall have that $^{11}/_{12}$ of the *denari* of the fourth are $^7/_{20}$ of the *denari* of the third and $^5/_{12}$ of the *denari* of the purse. *And you shall say, if $^{11}/_{12}$ of the fourth are $^7/_{20}$ of the *denari* of the third man and $\langle ^5/_{12} \rangle$ of the purse, what will all the *denari* of the fourth be? Where you divide by $^{11}/_{12}$, from which comes $^{21}/_{55}$ of the third and $^5/_{11}$ of the purse, and as much has the fourth man*. And this is done. And we shall make position that the second has a thing, that is, a quantity.[12] And because we have found that the first man has the $^4/_5$ of the *denari* of the purse and the $^{10}/_{20}$ of the *denari* of the second, the third man will thus have $^{13}/_{20}$ of quantity and $^4/_5$ of purse. And the fourth man, whom we have found that he has the $^{21}/_{55}$ of the third and $^5/_{11}$ of the purse. First you shall take the $^{21}/_{55}$ of that which the third has, of, that is, $^{13}/_{20}$ of quantity and $^4/_5$ of purse, which are $^{273}/_{1100}$ of quantity and $^{84}/_{275}$ of purse, and to the $^{84}/_{275}$ of purse you join $^5/_{11}$ of purse, they make $^{19}/_{25}$ of purse. And you shall say that the fourth man has the $^{273}/_{1100}$ of quantity and $^{19}/_{25}$ of purse. Now so as to know that which the first has you shall keep this way. The fourth and first with the purse have 5 times as much as the second. Therefore, if the second has 1 quantity, they will have with the purse 5 quantities. And we say that the fourth man has $^{273}/_{1100}$ of quantity and $^{19}/_{25}$ of purse. When to these has been joined a purse they make $^{273}/_{1100}$ ⟨of quantity⟩ and 1 purse $^{19}/_{25}$. And so much has the fourth with the *denari* of the purse and with the *denari* of the first. They should be 5 quantities, thus the first had, from the $^{273}/_{1100}$ ^of quantity^ and 1 purse $^{19}/_{25}$ until 5 quantities, where there are 4 quantities $^{827}/_{1100}$ less a purse $^{19}/_{25}$. And so much has the first. And in this way we have established that the first has 4 quantities $^{823}/_{1100}$ ⟨less 1 purse $^{19}/_{25}$⟩. The second has a quantity and the third has $^{13}/_{20}$ of quantity and $^{4}/_{5}$ of purse. And the fourth has $^{273}/_{1100}$ *quantities* and $^{19}/_{25}$ of quantity.[13] Now it is to be seen if the first and second *with the purse* have two times as much as the third. And you shall say, the first has 4 quantities $^{827}/_{1100}$ less 1 purse $^{19}/_{25}$. And the second has 1 quantity. These two amounts, having been joined with the *denari* of the purse, make 5 quantities $^{827}/_{1100}$ less $^{19}/_{25}$ of purse. And this is two times as much as the *denari* of the second,[14] that is, 2 times as much as $^{13}/_{20}$ quantity and $^{4}/_{5}$ of purse, which are $^{26}/_{20}$ of quantity and $^{8}/_{5}$ of purse. So 5 quantities $^{827}/_{1100}$ less $^{19}/_{25}$ of purse are equal to $^{26}/_{20}$ of quantity $^{8}/_{5}$ of purse, where you shall confront (*raguaglerai*[15]) the sides detracting on both sides $^{26}/_{20}$ of quantity and giving to each side $^{19}/_{25}$ of purse. You shall have that 3 quantities $^{1597}/_{1100}$ are equal to $^{59}/_{25}$ of purse. And in order not to have fractions, multiply each side by 1100. You shall have that 2897 quantities are equal to 2596 purses. Therefore, when the purse is worth 4897, the quantity is worth 2596. And for the first, who has 4 quantities $^{823}/_{1100}$ less a purse $^{19}/_{25}$, he will have 3717. And the second, whom we posited to have a quantity, will have 2596. And the third, whom we found to have $^{13}/_{20}$ quantity $^{4}/_{5}$ purse, will have 5605. And the fourth, whom we found to have $^{273}/_{1100}$ of quantity $^{4}/_{5}$ of purse, had 4366. And thus it has been made. The first has 3717. And the second has 2596. And the third has 5605. And the fourth 4366. And the purse had 4897. Which further, (fol. 271v) reduced to smaller numbers by 59: The first has 63, the second 44, the third 95, the fourth man 74. And the purse 83.

This accompanies a marginal calculation which, as already said, was written before the text. More precisely, as revealed by close attention to the organization of fol. 270^{v} (redrawn on the following page): the *statement* of the problem.

Four have *denari*, and walking on a road they found a purse with *denari*. [...]. It is asked how much each one had, and how many *denari* there were in the purse

was written first, and then Benedetto started calculating in a "margin" which in certain points invades the text column by more than 80%.



Fol. 270v, redrawn. Thick lines represent the problem statement, thin lines the procedure description (the first two lines the procedure of the previous problem).

The marginal calculation uses standard abbreviations for “first”, “second”, “third” and “fourth”, which for typographical convenience we may render α, ..., γ and δ. *Borsa* is abbreviated *b*.

The calculation starts “in the upper left corner”, in the best Göttingen manners; after that, however, it is no longer linear. The written text aids to clarify in which order the different parts are to be read; Benedetto himself obviously knew the order he was following.

At first (upper left corner of the “castelet”, as Benedetto calls it later) we find these four equations,[16]

(1) $$\gamma = {}^1/_2\,\alpha + {}^1/_2\,\beta + {}^1/_2 b,$$

(2) $$\delta = {}^1/_3\,\beta + {}^1/_3\,\Upsilon + {}^1/_3 b,$$

(3) $$\alpha = {}^1/_4\,\gamma + {}^1/_4\,\delta + {}^1/_4 b,$$

(4) $$\beta = {}^1/_5\,\alpha + {}^1/_5\,\delta + {}^1/_5 b,$$

derived from the initial conditions of the problem.

As a first step, to the right in the same chamber, ${}^1/_2\,\alpha$ is found from (3) and substituted in (1), yielding

(5) $$\gamma = {}^1/_8\,\gamma + {}^1/_8\delta + {}^1/_8 b + {}^1/_2\beta + {}^1/_2 b,$$

which is reduced to

(6) $${}^7/_8\,\gamma = {}^1/_8\delta + {}^1/_2\beta + {}^5/_8 b.$$

Next (2) is used to find ${}^1/_8\delta$, which is substituted into (6), leading to

(7) $${}^7/_8\gamma = {}^1/_{24}\beta + {}^1/_{24}\gamma + {}^1/_{24} b + {}^1/_2\beta + {}^5/_8 b,$$

which is reduced to

(8) $${}^5/_6\,\gamma = {}^{13}/_{24}\,\beta + {}^2/_3 b.$$

Division by ${}^5/_6$ transforms this into

(9) $$\gamma = {}^{13}/_{20}\,\beta + {}^4/_5 b\ .$$

In the next part of the calculation (next chamber downwards), ${}^1/_3\,\beta$ is found from (4) and inserted into (2), which leads to

(10) $$\delta = {}^1/_{15}\alpha + {}^1/_{15}\delta + {}^1/_{15} b + {}^1/_3\gamma + {}^1/_3 b.$$

This is reduced to

(11) $$^{14}/_{15}\delta = {}^{1}/_{15}\alpha+{}^{1}/_{3}\gamma+{}^{2}/_{5}b.$$

Now (3) is used to derive $^{1}/_{15}\,\alpha$, which is substituted into (11). That gives

(12) $$^{14}/_{15}\alpha = {}^{1}/_{60}\gamma+{}^{1}/_{6}0\alpha+{}^{1}/_{60}b+{}^{1}/_{3}\gamma+{}^{2}/_{5}b,$$

which reduces to

(13) $$^{11}/_{12}\delta = {}^{7}/_{20}\gamma+5/12b.$$

Division by $^{11}/_{12}$ reduces this to

(14) $$\delta = {}^{21}/_{55}\,\Upsilon+{}^{5}/_{11}b\ .$$

Now (the large chamber to the right) Benedetto shifts to the set of only two unknowns familiar from other problem solutions in the *Praticha* and from elsewhere, the *quantità* or *q* identified with β, and *borsa* or *b*, already in service. From (9) we get that

(15) $$\gamma = {}^{13}/_{20}\beta+{}^{4}/_{5}b,$$

and from (14) that

$$\delta = {}^{21}/_{55}.({}^{13}/_{20}q+{}^{4}/_{5}b)+{}^{5}/_{11}b,$$

whence

(16) $$\delta = {}^{273}/_{1100}q+{}^{19}/_{25}b.$$

Further, from the original condition behind (4) we know that

$$\delta+\alpha+b = 5.\beta,$$

whence

(17) $$\delta+\alpha+b = 5q.$$

This leads to

$$\alpha = 5q-({}^{273}/_{1100}q+{}^{119}/_{25}b),$$

that is

(18) $$\alpha = 4 8^{27}/_{100}q-1^{19}/_{25}b.$$

The values for α and β are now inserted into the original condition that gave rise to (1),

(19) $$2\gamma = \alpha+\beta+b,$$

from which follows

(20) $$2\gamma = {}^{5827}/_{1100}q - {}^{19}/_{2}5b,$$

that is,

(21) $$58^{27}/_{1100}q - {}^{19}/_{25}b = {}^{26}/_{20}\beta + {}^{8}/_{5}b$$

(even in the marginal calculation, ${}^{26}/_{20}$ appears without reduction). Addition and subtraction lead to

(22) $$31^{597}/_{1100}q = {}^{59}/_{25}b,$$

and after multiplication by 1100 "so as to avoid fractions"

(23) $$4897q = 2596b.$$

So (lower left chamber), if b is chosen to be 4897 (as Benedetto knows, the problem is indeterminate and allows this choice), q will be 2596. From (18) then follows that $\beta = 3717$; α is already known to be 2596, while γ is found for instance from (15) to be 5605, and δ (16) to be 4366; b is already known to be 4897.

But the problem is indeterminate. Benedetto's value for b was a *choice*, and all the other unknowns were found proportionally. Therefore, (lower right chamber) he reduces by the common factor 59, which gives him $\beta = 63$, $\beta = 44$, $\gamma = 95$, $\delta = 74$, $b = 83$. Since the coefficient 2596 in (23) arises as $59\cdot {}^{1100}/_{25} = 55\cdot 59$, it will have been obvious to try whether 59 is a common factor. Benedetto does not explain from where he has the number 59 but says to act "according to L[eonardo] P[isano]".

If we compare what is written in the marginal calculation with the text that explains the procedure it turns out that the latter contains a number of errors (indicated in the transcription in the appendix), confirming (if need should be) that the marginal calculation is made first, while the textual explanation is marred by typical copying errors. The marginal calculation thus presents us with a clear instance of *incipient symbolic algebra* involving five unknowns, antedating other known examples by a small century.[17]

A modern mathematician may (will) find this marvellous; after all, the manipulation of multiple variables is one of the characteristics of Viète's and Descartes' algebra, and thus an essential ingredient in the 17th-century transformation of the whole mathematical endeavour. Yet Benedetto apparently does not consider what he has done as epoch-making, even though the purpose for which he is writing would invite that – in a gift intended for a protector, it would be obvious to show the merit of what is offered indirectly by pointing out that something never made before is offered – certainly with due modesty, but Renaissance rhetoric no less than its modern counterpart was rich with tools for that.

As we shall see below, Benedetto *was* aware to have produced an innovation. A comparison with Fibonacci's *Liber abbaci* may tell us why he none the less seems to have considered this innovation marginal. In chapter 12 part 4 of Fibonacci's work [ed. Boncompagni 1857: 225; ed. Giusti 2020: 372] we find this problem:

> On four men and a purse.
>
> The first and the second with the purse have the double of the *denarii* of the third; and the second and the third the triple of the fourth, and then the third and the fourth the quadruple of the first, while the fourth and the first with the purse similarly have the quintuple of the second.

This is obviously the same problem, and the rest of Benedetto's *Praticha* leaves no doubt that Benedetto knew the *Liber abbaci*. Fibonacci goes on:

> The solution to this problem you will find by finding the ratio of the *denarii* of the purse to the *denarii* of the first in this way. Because the first and second with the purse have the double of the third, half of the *denarii* of the first and second and the purse is as much as the *denarii* of the third man. Similarly from the other propositions you will have that $^1/_3$ of the second and third man and of the purse is as much as the *denarii* of the fourth man, and $^1/_4$ of the third and fourth man and of the purse is the quantity of the *denarii* of the first, and $^1/_5$ of the *denarii* of the fourth and first man and of the purse is the quantity of the *denarii* of the second. And because $^1/_2$ of the first and second and of the purse is the quantity of the third, the third part of the first and second and purse, that is $^1/_6$ of them, is $^1/_3$ of the third man. Commonly (*comuniter*) are joined $^1/_3$ of the *denarii* of the second and purse: then will $^1/_6$ of the first and $^1/_2$ of the second and of the purse be as much as $^1/_3$ of the second and third and of the purse. But $^1/_3$ of the second and third and of the purse is the quantity of the *denarii* of the fourth man; hence $^1/_6$ of the first and $^1/_2$ of the second and of the purse are the quantity of the *denarii* of the fourth man. Therefore $^1/_4$ of $^1/_6$ of the *denarii* of the first, that is, $^1/_{24}$, and $^1/_4$ of $^1/_2$, thus $^1/_8$ of the *denarii* of the second and of the purse, are $^1/_4$ of the *denarii* of the fourth man. Commonly are added $^1/_4$ of the third and of the purse: then $^1/_{24}$ of the first with $^1/_8$ of the second and with $^1/_4$ of the third and $^3/_8$ of the

> purse will be as much as $^1/_4$ of the *denarii* of the third and fourth and of the purse. But $^1/_4$ of the third man and the fourth and of the purse is the quantity of the first. Therefore $^1/_{24}$ of the first and $^1/_8$ of the second and $^1/_4$ of the third and $^3/_8$ of the purse are as much as the *denarii* of the first. Then their fifth part, that is $^1/_{120}$ of the first and $^1/_{40}$ of the second and $^1/_{20}$ of the third and $^3/_{40}$ purse, are $^1/_5$ of the *denarii* of the first. Commonly are added $^1/_5$ of the fourth man and the purse: then $^1/_{120}$ of the first and $^1/_{40}$ of the second and $^1/_{20}$ of the third and $^1/_5$ of the fourth and $^{11}/_{40}$ of the purse will be as much as $^1/_5$ of the fourth man and the first and of the purse. [...]

The omission [...] is as long as the part that was translated. It leads to

> Hence $^{79}/_{600}$ and $^1/_{150}$ of the first, that is $^{83}/_{600}$ of the same, with $^1/_{25}$ of the purse, are $^{29}/_{200}$ of the purse. Commonly are taken away $^1/_{25}$ of the purse. Remain $^{83}/_{600}$ of the first, as much as $^{21}/_{200}$ of the purse. Then two numbers should be found so that $^{83}/_{600}$ of the first are $^{21}/_{200}$ of the second, they will be 63 and 83. Then if the first man has 63, the purse is 83. [...].

If we admit the identity of "the *denari* of the first/first man", "the quantity of the *denari* of the first man", "the quantity of the first man" and "the first man", this is rhetorical algebra with five unknowns according to Heeffer's definition (see note 2). If we insist on fully consistent and not only unmistakeable naming it may not be, but the difference is scant, and would hardly have been thought of at the time.[18]

But even here, there are traces in the text that Fibonacci *described* a procedure performed by other means. Several errors are of the same type as those found in Benedetto's description: "150" instead of "$^1/_{150}$ primi" and "denariis secondi" instead of "denariis primi". Both are listed as belonging to the hypothetical manuscript ω in [Giusti 2020: 373], which Giusti supposes to be an already imperfect copy of Fibonacci's original from which all extant manuscripts should derive (there are many of these "ω errors" throughout the work).

A discovery made by Giusti [2017] rules out this explanation of the errors. In one *Liber abbaci* manuscript,[19] chapter 12 is an earlier version, and since we know of no intermediate versions presumably the first version from 1202. In any case it is clearly earlier, and most of the "ω errors" are also found in this manuscript. This means that these errors were in a master copy from which Fibonacci prepared the "1228" version that we know – that is, they were made by Fibonacci himself, even though they are evidently the kind of errors that a copyist would make.

So, returning to our topic, Fibonacci when describing the procedure in rhetorical algebra copied from somewhere, and with high probability from his own calculation. This *could* be a solution by rhetorical algebra made separately, but it might also be a solution by means of the kind of line diagrams of which he made use elsewhere when solving problems by means of "the rule of ratios".[20]

In any case, Fibonacci's solution by means of rhetorical algebra in the version of the *Liber abbaci* at Benedetto's disposal was more complicated than Benedetto's own solution by means of rudimentary symbolic algebra. As we shall see, Benedetto was aware that his method was new and not the same as that of Fibonacci; but he may also have acknowledged that the innovation was marginal.

Sharpening the tool

The evidence that Benedetto was aware of having innovated is found in later problems in the *Praticha* of the type "purchase of a horse" (two purse problems following immediately after the one we have examined make use of only two unknowns, *quantità* and *borsa*, after which one on fol. 272^{r} is solved *per chonsideratione*, that is, by means of arithmetical arguments).

The first horse problems are also solved via arithmetical arguments or (most of them) by means of two algebraic unknowns, here *quantità* and *chavallo* (the latter abbreviated *c* in marginal calculations). Then, on fol. 277^{r}, comes this:[21]

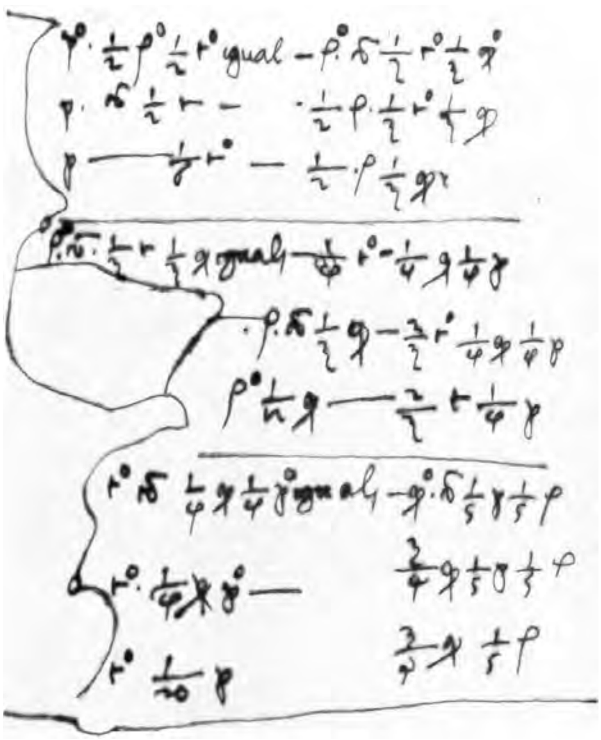
> (fol. 277r)Four men have *denari* and want to buy a horse, and no one has so many *denari* that he can buy it. The first says to the second and the third, if you give me $^{1}/_{2}$ of your *denari*, with mine I shall buy the horse. The second says to the third and fourth man, if you give me the $^{1}/_{3}$ of your *denari*, with mine I shall buy the horse. The third man says to the fourth and the first, if you give me the $^{1}/_{4}$ of your *denari*, I shall buy the horse. Further, the fourth man asks the first and the second for the $^{1}/_{5}$ of their *denari* and says to buy the horse. It is asked, how many *denari* each one had, and what the horse was worth.
>
> Even though there are many ways to solve such cases I shall take the most convenient, or let us say the least tedious. || That is that you shall say, we propose that the first with the half of the *denari* of the second and of the third man has a horse. And we say that the second with the third of the *denari* of the third and fourth man buy the horse. So the first with the $^{1}/_{2}$ of the *denari* of the second and third man has as much as the second has with $^{1}/_{3}$ of the third and fourth man. From there you

will see confronting (*raguagliando*), that is, detracting first on each side (*parte*) ${}^{1}/_{2}$ of the *denari* of the second, and we shall have that the *denari* with ${}^{1}/_{2}$ of the *denari* of the third man are as much as ${}^{1}/_{2}$ of the *denari* of the second with ${}^{1}/_{3}$ of the third and fourth man. And then you remove from each side ${}^{1}/_{3}$ of the *denari* of the third man, and we shall have that the first man with ${}^{1}/_{6}$ of the *denari* of the third man is as much as ${}^{1}/_{2}$ of the *denari* of the second. And take note of this (*nota*). [...]

This text, until the point marked ||, was written first. Then Benedetto starts calculating in the margin, at first in these steps (once more using standard abbreviations for "first", "second", "third" and "fourth", here again rendered α, β and γ):[22]

$$\alpha + {}^{1}/_{2}\,\beta + {}^{1}/_{2}\gamma = \beta + {}^{1}/_{3}\gamma + {}^{1}/_{3}\delta$$
$$\alpha + {}^{1}/_{2}\gamma = {}^{1}/_{2}\,\beta + {}^{1}/_{3}\gamma + {}^{1}/_{3}\delta$$
$$\alpha + {}^{1}/\gamma = {}^{1}/_{2}\,\beta + {}^{1}/_{3}\delta$$

$$\beta + {}^{1}/_{3}\gamma + {}^{1}/_{3}\gamma = \gamma + {}^{1}/_{4}\delta + {}^{1}/_{4}\alpha$$
$$\beta + {}^{1}/_{3}\delta = {}^{2}/_{3}\gamma + {}^{1}/_{4}\delta + {}^{1}/_{4}\alpha$$
$$\beta + {}^{1}/_{12} = {}^{2}/\gamma + {}^{1}/_{4}\alpha$$



The first part of the calculations on fol. 277r, redrawn.

The structure of the marginal calculation is similar to that of the previous example – divided into sections, the first of these (redrawn here) written close to the margin and not occupying much of the text column, those written later then forced further into it. There is no need to say more about this.

The calculation was thus made first even this time, and the describing text written afterwards. In what follows, we see a sharpening of Benedetto's conceptual apparatus, speaking explicitly about the (reduced) equations and giving a name to the isolation of one unknown. We may take this as evidence that Benedetto is not only solving just these two problems in an innovative way but also gradually shaping a new tool:

> And then you shall say, we have said that the second man with $^1/_3$ of the *denari* of the third and fourth man has as much as the third man with $^1/_4$ of the *denari* of the fourth and first man. Where the *denari* of the second man with $^1/_3$ of the *denari* of the third and fourth man are as many as are the *denari* of the third man with $^1/_4$ of the *denari* of the fourth and first (fol. 277v)man. Where confronting the sides you take away $^1/_3$ of the *denari* of the third man, and you shall have the *denari* of the second with the $^1/_3$ of the *denari* of the fourth to be the $^2/_3$ of the *denari* of the third man with $^1/_4$ of the *denari* of the fourth and first man. And then on each side you take away $^1/_4$ of the fourth man. And you shall have that the *denari* of the second man with $^1/_{12}$ of the fourth man are as much as $^2/_3$ of the third man with $^1/_4$ of the *denari* of the first man. Which you still take note of. And in this way you may confront[23] the other positions. But with these 2 you can solve. And if you want the other equations,[24] you may do as you see on the previous page, there the equation of the third is.[25] Now to our subject-matter (*materia*). We have made that the *denari* of the first with $^1/_6$ of the *denari* of the third man are $^1/_2$ of the *denari* of the second and $^1/_3$ of the *denari* of the fourth man. And we also have that the second with $^1/_{12}$ of the fourth man are as much as the $^2/_3$ of the *denari* of the third man and $^1/_4$ of the *denari* of the first. Therefore, $^1/_4$ of the *denari* of the first is to be brought apart from the *denari* of the others. You shall keep this way, we have that the *denari* of the first and $^1/_6$ of the *denari* of the third man are as much as the $^1/_2$ of the *denari* of the second and $^1/_3$ of the *denari* of the fourth man, where from both sides you take away $^1/_6$ of the *denari* of the third man. And you shall have the *denari* of the first to be $^1/_2$ of the *denari* of the second and $^1/_3$ of the *denari* of the fourth less $^1/_6$ of the *denari* of the third man. Therefore $^1/_4$ of the *denari* of the first man are $^1/_8$ of the *denari* of the second and $^1/_{12}$ of the *denari* of the *denari* of the fourth and less $^1/_{24}$ of the *denari* of the third man.[26] And that you shall join to $^2/_3$ of the *denari* of the third man, and you shall have $^5/_8$ of the *denari* of the third man and $^1/_8$ of the second and $^1/_{12}$ of the fourth man. Therefore, you shall say that the *denari* of the second with $^1/_{12}$ of the *denari* of the fourth are as much as $^5/_8$ of the *denari* of the third man and $^1/_{12}$ of the *denari* of the fourth man and $^1/_8$ of the *denari* of the second. Therefore, from each side you shall take away $^1/_8$ of the second man and $^1/_{12}$ of the fourth man. You shall have that $^5/_8$ of the *denari* of the second are as much as $^5/_8$ of the *denari* of the third man. Now this is known, ^you shall say^, if the second man should have 5, then the third would have 7. And having had this insight (*lume*), and we shall make position that

> the second man had 5 quantities, it follows that the third man would have 7 quantities. [...].

From here onward, in the margin and as described in the text, Benedetto makes (symbolic respectively rhetorical) algebra with two unknowns only, and we do not need to follow him.

The next problem (fol. 278^r), with a similar structure but only three participants, is borrowed from the *Liber abbaci* [ed. Boncompagni 1857: 242; Giusti 2020: 396]. Fibonacci solves it with not too intricate arithmetical arguments, Benedetto instead by means of rhetorical algebra involving the unknowns *quantità* and *chavallo*. The rhetorical argument is accompanied by marginal symbolic calculations that may as well have been written after as before the rhetorical text. In any case, there are some corrections in the statement of the problem (an omission of the intention to buy a horse), which was copied from Fibonacci, but none in the solution.

Then, on fol. 278^v, comes another problem taken over from Fibonacci [B243; G397], and here Benedetto again brings his new method into play. He *might* have expressed Fibonacci's quasi-algebraic procedure within the new framework and does so in the initial steps – not necessarily copying, these are simply the obvious first steps. Then, however, the two solutions diverge.

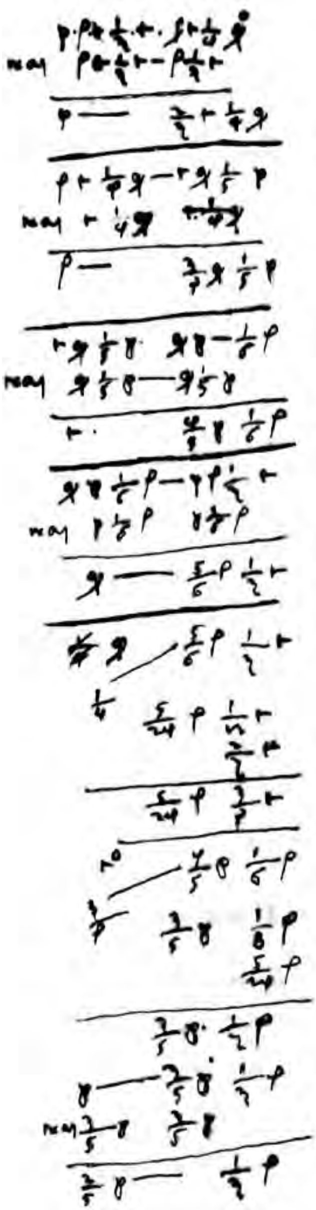
> Four have *denari* for which they want to buy a horse, and none of them has so many *denari* that he can buy it. The first and the second say to the third man, if you give us the $^1/_3$ of your *denari*, we shall buy the horse. The second and third man say to the fourth man, if you give us the $^1/_4$ of your *denari* we shall buy the horse. The third and fourth man say to the first, if you give us the $^1/_5$ of your *denari*, with ours we shall buy the horse. The fourth and first man say to the second, if you give us the $^1/_6$ of your *denari* we shall buy the horse. It is asked, how much each one had, and what the horse was worth. We shall do it by equation.[27] Where you shall say, the first and second with $^1/_3$ of the third buy the horse. And the second and third man with $^1/_4$ of the fourth man buy the horse. Thus the *denari* of the first and second with $^1/_3$ of the *denari* of the third man are as much as are the *denari* of the second and third man with $^1/_4$ of the *denari* of the fourth. Where confronting the sides, taking away from each side the *denari* of the second and $^1/_3$ of the *denari* of the third, we shall have that the *denari* of the first are as much as $^2/_3$ of the *denari* of the third man and $^1/_4$ of the *denari* of the fourth man. And mark[28] this. Then you shall say, the second and third man with $^1/_4$ of the *denari* of the fourth man buy the horse. And the third and fourth man with $^1/_5$ of the *denari* of the first buy a horse. So the *denari* of the second and third

man with $^1/_4$ of the *denari* of the fourth man are as much as the *denari* of the third and fourth man with $^1/_5$ of the *denari* of the first. Therefore take away from each side the *denari* of the third and $^1/_4$ of the *denari* of the fourth man, and we shall have that the *denari* of the second are $^3/_4$ of the *denari* of the fourth and $^1/_5$ of the *denari* of the first. And then, going on, you shall say that the third and fourth man with $^1/_5$ of the *denari* of the first buy the horse. And the fourth and first with $^1/_6$ of the *denari* of the second buy the horse. It therefore follows that the *denari* of the third and fourth man with $^1/_5$ of the *denari* of the first are as much as the first and fourth with $^1/_6$ of the *denari* of the second. Where, confronting the sides, taking away on each side the *denari* of the fourth man and $^1/_5$ of the first, we shall have that the third man has the $^4/_5$ of the first and $^1/_6$ of the second. And mark this. And thus you shall do for the fourth man, saying, the first and fourth with the $^1/_6$ of the *denari* of the second buys the horse. The first and second with the $^1/_3$ of the *denari* of the third buy the horse. Therefore the fourth and first with the $^1/_6$ of the *denari* of the second have as much as the first and second with $^1/_3$ of the *denari* of the third man. Therefore confronting, the sides, taking away on each side the *denari* of the first and $^1/_6$ of the *denari* of the second, we shall have that the *denari* of the fourth are $^5/_6$ of the *denari* of the second and $^1/_3$ of the *denari* of the third. And of that has been taken note. And you shall begin at the first equation,[29] saying, the *denari* of the first are the $^2/_3$ of the *denari* of the third man and $^1/_4$ of the fourth man. Therefore it has to be known what $^1/_4$ of the *denari* of the fourth are. From the others, however, we have found that the *denari* of the fourth man are the $^5/_6$ of the second and $^1/_3$ of the third man, where the $^1/_4$ of the *denari* of the fourth man are as much as the $^5/_{24}$ of the *denari* of the second and $^1/_{12}$ of the *denari* of the third. Where to the $^2/_3$ of the *denari* of the third man you join $^1/_{12}$ of the *denari* of the third and $^5/_{24}$ of the second, they make $^3/_4$ of the third and $^5/_{24}$ of the *denari* of the second. And then bring the $^3/_4$ of the third apart from the others, saying, the third man has the $^4/_5$ of the first and $^1/_6$ of the second, where the $^3/_4$ of the third man are the $^3/_5$ of the first and $^3/_{24}$ of the second. And you shall join to $^5/_{24}$ of the second $^3/_5$ of the first and $^3/_{24}$ of the second, they make $^3/_5$ of the first and $^1/_3$ of the second, and we shall have made that the *denari* of the first are as much as $^3/_5$ of the first and $^1/_3$ of the second. Therefore you shall detract on both sides the *denari* of the first, you shall have that $^2/_5$ of the *denari* of the first are $^1/_3$ of the *denari* of the second. That is, that the $^2/_5$ of the *denari* of the first are as much as the $^1/_3$ of the *denari* of the second. Thus, if the first should have 5, the second would have 6. Let us now try the others. You shall say that the third has as much as the $^4/_5$ of the first and the $^1/_6$ of the second. Therefore, the $^4/_5$ of the first and $^1/_6$ of the second are 5. And so much would he have. And the fourth has $^5/_6$ of the second and $^1/_3$ of the third, where the $^5/_6$ of the second are 5 and $^1/_3$, and the $^1/_3$ of the third man are $1^2/_3$, (fol. 279r)which all make $6^2/_3$. And thus it is done, the first has 5 and the second 6 and the third 5 and the fourth $6^2/_3$. Which, so as not to have fractions, multiply all by 3. And you shall have the first 15, the second 18, and the third man 15, and the fourth man 20. And so as to know what the horse is worth, you shall join 15 of the first and 18 of the second, they make 33. To these joined the $^1/_3$ of the *denari* of the third man, that is, of 15, they make 38. And as much is worth the horse. And thus the first had 15, the second 18, and the third had 15, and the fourth man had 20. And the horse was worth 38.

Now the technique is mature. Firstly we observe that Benedetto no longer falls for the temptation to shift to the traditional two unknowns, he uses the four unknowns (the price of the horse does not enter in the algebraic manipulations) until the very end. Secondly, the marginal calculation is extremely neat. It fills only a narrow column in the margin and does not go into the text column, but the corrected errors in the describing text still suggest that the marginal calculations were made first.

It is to be observed, however, that from this point onward, the manuscript contains no more invasive marginal calculations made before the text was written – those that intrude can be seen to be made in already prepared triangular or rectangular spaces. That in not very significant from fol. 300^r onward: from there, the substance is taken over from Fibonacci or earlier prestigious abbacus authors (due credit given), and whatever marginal material was to be inserted was known to Benedetto from the originals. Even fols 288^v– 299^v, containing "erratic" recreational problems "for which the rules vary" may not be informative. But after the present problem there are still ten sheets (twenty pages) with horse problems, where marginal calculations could be expected, and all we find in the margin are scattered brief notes extracted from the text, in the style "*t*^{o} 12 *q* mê 6 *ca*" ("the third, 12 *quantità* less 6 horses" – thus on fol. 282^v). It looks as if Benedetto has now decided to be a clean writer and make his draft work separately.[30]

It is therefore quite possible that Benedetto already at this point made his symbolic calculations on a different support – a scrap sheet of paper, a slate,[31] or whatever – and then copied them neatly into the margin, before or after writing the textual description of the procedure. However that may be, it will be informative to look at Benedetto's symbolic calculation on the next page (redrawn and transcribed from the manuscript – the final summary is left out as uninteresting for the present purpose).



$$\alpha+\beta+\tfrac{1}{3}\gamma = \beta+\gamma+\tfrac{1}{4}\delta$$

detract $\beta+\tfrac{1}{3}\gamma = \beta+\tfrac{1}{3}\gamma$

$$\alpha = \tfrac{2}{3}\gamma+\tfrac{1}{4}\delta$$

$$\beta+\gamma+\tfrac{1}{4}\delta= \gamma+\text{ä}+\tfrac{1}{5}\delta$$

detract $\gamma+\tfrac{1}{4}\delta= \gamma+\tfrac{1}{4}\delta$

$$\beta = \tfrac{3}{4}\delta+\tfrac{1}{5}\alpha$$

$$\gamma+\delta+\tfrac{1}{5}\alpha = \delta+\alpha+\tfrac{1}{6}\beta$$

detract $\delta+\tfrac{1}{5}\alpha = \delta+\tfrac{1}{5}\alpha$

$$\gamma = \tfrac{4}{5}\alpha+\tfrac{1}{6}\beta$$

$$\delta+\alpha+\tfrac{1}{6}\beta = \alpha+\beta+\tfrac{1}{3}\gamma$$

detract $\alpha+\tfrac{1}{6}\beta = \alpha+\tfrac{1}{6}\beta$

$$\delta= \tfrac{5}{6}\alpha+\tfrac{1}{3}\gamma$$

δ $\quad$ $\tfrac{5}{6}\alpha+\tfrac{1}{3}\gamma$

$\tfrac{1}{4}$

$\tfrac{5}{24}\beta+ \tfrac{1}{12}\gamma$

$\tfrac{2}{3}\gamma$

$$\tfrac{5}{24}\beta+ \tfrac{3}{4}\gamma$$

γ $\quad$ $\tfrac{4}{5}\alpha+\tfrac{1}{6}\beta$

$\tfrac{3}{4}$

$\tfrac{3}{5}\alpha+\tfrac{1}{8}\beta$

$\tfrac{5}{24}\beta$

$$\tfrac{3}{5}\alpha+\tfrac{1}{3}\beta$$

$$\alpha = \tfrac{3}{5}\alpha+\tfrac{1}{3}\beta$$

detract $\tfrac{3}{5}\alpha = \tfrac{3}{5}\alpha$

$$\tfrac{2}{5}\alpha = \tfrac{1}{3}\beta$$

Redrawn marginal calculation from fol. 278v (left) with transcription (right)

The previous two symbolic calculations evidently sufficed for Benedetto to solve the problems, but it would not have been easy for a reader to follow the thread of his reasoning through his "castelets" without his explanatory text. The present, instead, should function on its own for the reader – at least for a reader ready to grasp its principles. So, at this point Benedetto has really reached the stage of "public" (as opposed to "private") symbolic algebra, albeit only for a specific type of problem. So, while in the two preceding instances Benedetto's draft calculations may be claimed to be not quite as developed as those of Buteo (which, however, were not drafts but prepared for print), the present one-column organization is quite at the same level and very similar in principle.[32]

A related problem follows on fol. 279^{r}, dealing with five men who in groups of three ask the next one in the cycle for, respectively, $^{1}/_{4}$, $^{1}/_{5}$, $^{1}/_{6}$, $^{1}/_{7}$ and $^{1}/_{8}$ of their money so as to be able to buy the horse.[33] Even this case, Benedetto says, will be made "by equation", and he shows how the first reduced equation is to be constructed. For the others he refers to "the teaching made below", and a space large 11.5 centimetres and high 9 centimetres is indeed left blank there; unfortunately, it has not been filled by calculations, but at least we see that Benedetto thought that his symbolic calculations would be preferable to a verbal description.

So, now Benedetto has developed an orderly notation for symbolic algebraic solution of linear problems with up to five unknowns (and with nothing preventing more); and he has shown that he feels confident that it can be useful to, and used by readers. And then he almost stops. In the remaining 10 folios of the chapter there are 24 more horse problems (two instead dealing with the purchase of a goose respectively a hare). Two (fols 280^{r} and 280^{v}), about an impossible problem and a possible counterpart, are drawn from the *Liber abbaci* [ed. Boncompagni 1857: 251*f*; ed. Giusti 2020: 408*f*] and summarize Fibonacci's method. One (fol. 285^{r}) is told to be a repetition and discarded as such (Benedetto obviously discovers after having written the statement); 19 are solved by means of the two unknowns *quantità* and *chavallo* (respectively *ocha*, "goose", or *lepra*, "hare"). One of these (fol. 287^{r}) does not go through the calculations but merely says

> you shall hold the way given before, that is, you shall make position that the first has a quantity, and following in order the way given before, and we shall have that the first man will have 74 florins [...],

thus confirming that Benedetto now makes use of a separate medium for his calculations – the margin is empty.

In a problem on fol. 286^v dealing with five men and five horses (the prices of which differ by known amounts) we find the similar

> where in the way given before, making and observing well, you shall have that the first man had 1589 florins [...].

Once again, there is no further explanation, nor any annotation in the margin. Here, given the complexity of the problem, the way referred to *might* be "by equation"; but we cannot be sure – in the impossible problem taken over from Fibonacci we also find a reference to "the way given before", and there it appears to refer to what has been set out before *by Fibonacci* but not by Benedetto.

But it *might* still refer to the method "by equation", since this method is actually used in a simpler way on fol. 282^r in a problem about four men, of whom the first asks the second for $^1/_2$ of his *denari* and the third for $^1/_3$ of his (etc.). No detailed calculations are presented, but when the first manipulations of the rhetorical equations show that "$^4/_5$ of the first are as much as $^4/_5$ of the second" we find the usual phrase "and take note of this" (*e questo nota*), and a corresponding note in the margin; the next two simplified equations are also written in the margin, though without "take note". Given that the arguments are relatively simple we cannot know whether Benedetto had recourse to separate symbolic calculations or made all arguments directly in words.

After that there is no occasion to make use of the technique. If we had expected to find an important step toward the creation of modern mathematics we will probably be disappointed. True, Benedetto's *Pratica* was sufficiently respected to be copied (two copies survive, both incomplete; more are likely to have existed). None the less, I have not discovered inspiration from this innovative aspect of Benedetto's work. In [Høyrup 2019: 151–156] I have discussed the reasons for this failure to produce a breakthrough, and shall not repeat the arguments – just sum up that a breakthrough which nobody has a reason to find interesting is likely to become a dead-end and no genuine breakthrough. If the one producing it sees what has been made as nothing but a marginal innovation and does nothing to convince others that they should listen, the probability that the idea will be taken up dwindles to zero.

Appendix: Transcription of the Tuscan passages on which the translations are based

Abbreviations have been expanded (which at times involves some guesswork as to the intended orthography, which in fully written words is not too uniform); a distinction between *u* and *v* has been introduced, and accents and apostrophes have been added, mostly in agreement with present-day Italian habits (but also indicating, for instance, that an *a* in the text is a verb and corresponds to modern Italian *ha* (thus *à*), to distinguish it from the preposition *a*). A few full stops are indicated by Benedetto (not always where we would expect them), but the present punctuation replaces them.

Words in ⟨ ⟩ have been forgotten by Benedetto and are restored from his preceding symbolic calculations after control that they are presupposed in what follows; words in ^ ^ were at first forgotten and then inserted by Benedetto between the lines. The two passages marked * * were initially omitted and then written in the margin. Crossed-out words have been deleted by Benedetto. ¿ ? indicates that I am in doubt about the word in question. For the sake of layout convenience, the horizontal fraction lines of the manuscript have been replaced by slashes.

(fol. 270v)Quatro ànno denari e andando per una via trovarono una borsa di denari. Dichono el primo e'l secondo al terço, se'ttu ci dai la borsa noi aremo 2 chotanti di te. Dice il secondo e terço huomo al quarto, se noi avessimo e denari della borsa noi aremo 3 chotanti de te. Dice el terço e quarto al primo, se noi avessino e denari della borsa noi aremo 4 chotanti di te. Dice el quarto e il primo al secondo, se'ttu ci dai e denari della borsa noi aremo 5 cotanti di te. Adimandosi quanto aveva ciaschuno, e quanti denari erano nella borsa. In tale modo faremo che primo dirai, el primo e secondo cho' denari della borsa dichono d'avere 2 chotanti che'l terço huomo. Onde el terço huomo da'sse aveva il $^1/_2$ di ciò che à il primo e il secondo e della borsa. E questo segnia. E dipoi dirai, el secondo e terço huomo cho'lla borsa ànno 3 chotanti del quarto, Adunque el quarto huomo aveva il $^1/_3$ di ciò che à il primo e secondo e della borsa. E anchora questo segnia. E dipoi dirai, il terço e quarto huomo cho'lla borsa aveva 4 chotanti del primo, e però il primo da'sse medesimo aveva il $^1/_4$ di ciò che ànno il terço e quarto e della borsa. E questo segnia. E dipoi dirai, el quarto e primo ànno cho'lla borsa 5 chotanti del secondo, che il secondo arà il $^1/_5$ di ciò che ànno fra il primo e quarto huomo e della borsa. È questo segnato e tu arrecherai e denari del terço a una chomparatione. E dirai, e denari del terço huomo sono quanto il $^1/_2$ de' denari del primo e del secondo e dela borsa. Onde è da sapere quanto sono el $^{1}/_{2}$ de' denari del primo che abbiamo ¿chopagnato? che' denari del primo sono il $^{1}/_{4}$ de' denari del terḍo e quarto huomo e della borsa, e il $^{1}/_{2}$ de' denari del primo sieno $^{1}/_{8}$ de' denari del terço e quarto e della borsa. Onde dirai che' denari del terço sieno quanto il $^{1}/_{2}$ de' denari del secondo e della borsa e quanto $^{1}/_{8}$ de' denari del terço e quarto e della borsa. Onde leverai $^{1}/_{8}$ de' denari del terço e agugnerai $^{1}/_{8}$ de borsa a $^{1}/_{2}$ borsa. E aremo che $^{7}/_{8}$ de' denari del terço sono $^{1}/_{8}$ de' denari del quarto e $^{1}/_{2}$ de' denari del secondo e $^{5}/_{8}$ di borsa. E dipoi (fol. 271r) $^{1}/_{8}$ de' denari del quarto è da vedere quanto sono, che diciamo che tutti e denari del quarto huomo sono $^{1}/_{3}$ del secondo e del terço huomo e della borsa. Onde $^{1}/_{8}$ de' denari del quarto huomo sieno $^{1}/_{24}$ de' denari del secondo e del terço e della borsa. Onde leverai de' $^{7}/_{8}$ de' denari del terço $^{1}/_{24}$ de' denari del terço huomo e sopra $^{5}/_{8}$ di borsa porrai $^{1}/_{24}$ di borsa e sopra $^{1}/_{2}$ del secondo porrai $^{1}/_{24}$ de' denari del secondo, e aremo che $^{5}/_{6}$ de' denari del terço huomo sono quanto e $^{2}/_{3}$ della borsa e $^{13}/_{24}$ de' denari del secondo. E dirai, se $^{5}/_{6}$ de' denari del terço huomo sono quanto e $^{2}/_{3}$ della borsa e $^{13}/_{24}$ del secondo, tutti i denari del terço quanto sieno. Dove partirai $^{2}/_{3}$ de' denari della ~~p~~ borsa e $^{13}/_{24}$ de' denari del secondo in $^{5}/_{6}$, vien'ne $^{4}/_{5}$ de' denari della borsa e $^{13}/_{20}$ de' denari del secondo. È questo notato. E tu in nello medesimo modo farai e' denari del quarto. Bene è vero che sança si potrebbe fare, ma dappoi che nel castelluccio è fatto perseguirà quell'ordine. Dirai, il quarto à il $^{1}/_{3}$ de' denari del secondo e del terço e della borsa. Dove arrecha el $^{1}/_{3}$ de' denari del secondo a parte degli altri, e dirai, il secondo à quanto $^{1}/_{5}$ del primo e del quarto e della borsa. Onde il $^{1}/_{3}$ de' denari del secondo sieno $^{1}/_{15}$ de' denari del primo e del quarto e della borsa. Dove de' denari del quarto leverai $^{1}/_{15}$ e agugnerai a $^{1}/_{3}$ di borsa $^{1}/_{15}$ di borsa e aremo che $^{14}/_{15}$ del quarto sono quanto il $^{1}/_{15}$ del primo e $^{1}/_{3}$ del terço e $^{2}/_{5}$ di borsa. Dipoi $^{1}/_{15}$ del primo sieno $^{1}/_{60}$ del terḍo e $^{1}/_{60}$ del quarto e della borsa, dove de' $^{14}/_{15}$ de' denari del quarto leva $^{1}/_{60}$ e sopra $^{1}/_{3}$ de' denari del terḍo huomo agugni $^{1}/_{60}$ de' denari del terço huomo e sopra e $^{2}/_{5}$ de' denari della borsa poni $^{1}/_{60}$ di borsa, e arai che $^{11}/_{12}$ de' denari del quarto sono $^{7}/_{20}$ de' denari del terço e $^{5}/_{12}$ de' denari della borsa. *E dirai, se $^{11}/_{12}$ del quarto sono $^{7}/_{20}$ de' denari del terço huomo e ?$^{5}/_{12}$? della borsa, che sieno tucto e denari del quarto, dove partirai in $^{11}/_{12}$, vien'ne $^{21}/_{55}$ del terço e $^{5}/_{11}$ della borsa, e tanto à il quarto huomo.* È questo fatto. E noi faremo positione ch'el secondo abbia una chosa cioè una quantità. E perché noi abbiamo trovato ch'el terço huomo à e $^{4}/_{5}$ de' denari della borsa e gli $^{13}/_{20}$ de' denari del secondo, arà dunque il terço huomo $^{13}/_{20}$ di quantità e $^{4}/_{5}$ di borsa. E il quarto huomo che abbiamo trovato che à e $^{21}/_{55}$ del terço e ^{5}/ 11 della borsa. Primo piglierai e $^{21}/_{55}$ di ciò che à il terço di cioè $^{13}/_{20}$ de quantità e $^{4}/_{5}$ di borsa, che sono $^{273}/_{1100}$ di quantità e $^{84}/_{275}$ di borsa, e agli $^{84}/_{275}$ di borsa agugnerai $^{5}/_{11}$ di borsa, fanno $^{19}/_{25}$ di borsa. E dirai ch'el quarto huomo abbia e $^{273}/_{1100}$ di quantità e $^{19}/_{25}$ di borsa. Ora per sapere quel che à il primo terrai quel modo. El quarto e primo cho'lla borsa ànno 5 chotanti del secondo. Onde se'l secondo à 1 quantità, egli aranno colla borsa 5 quantità. E noi diciamo ch'el quarto huomo à $^{273}/_{1100}$ di quantità e $^{19}/_{25}$ di borsa. A quelli agunto una borsa fanno $^{273}/_{1100}$?di quantità? e 1ª borsa $^{19}/_{25}$. E tanto à il quarto cho' denari della borsa e cho' denari del primo. Debbono essere 5 quantità, adunque il primo aveva da e $^{273}/_{1100}$ ^di quantità^ e 1ª borsa $^{19}/_{25}$ infino in 5 quantità, che v'è 4 quantità $^{827}/_{1100}$ meno una borsa $^{19}/_{25}$. E tanto à il primo. E chosì abbiamo ordinato ch'el primo à 4 quantità $^{823}/_{1100}$ ⟨meno 1 borsa $^{19}/_{25}$⟩. El secondo à una quantità e il terço à $^{13}/_{20}$ di quantità e $^{4}/_{5}$ di borsa. E il quarto huomo à $^{273}/_{1100}$ *quantità* e $^{19}/_{25}$ di quantità.[34] Ora è da vedere s'el primo e secondo *cho'lla borsa* ànno 2 chotanti ch'el terço. E dirai, il primo à 4 quantità $^{827}/_{1100}$ meno 1ª borsa $^{19}/_{25}$. E el secondo à 1ª quantità. Ragunto queste 2 some cho' denari della borsa faranno 5 quantità $^{827}/_{1100}$ meno $^{19}/_{25}$ di borsa. E questo è duo chotanti che' denari del secondo[35], cioè, 2 chotanti di $^{13}/_{20}$ quantità e $^{4}/_{5}$ di borsa, che sono $^{26}/_{20}$ di quantità e $^{8}/_{5}$ di borsa. Adunque 5 quantità $^{827}/_{1100}$ meno $^{19}/_{25}$ di borsa sono iguali a $^{26}/_{20}$ di quantità $^{8}/_{5}$ di borsa, dove raguaglerai le parti traendo da ogni parte $^{26}/_{20}$ di quantità e dando a ogni parte $^{19}/_{25}$ di borsa. Arai che 3 quantità $^{1597}/_{1100}$ sono iguali a $^{59}/_{25}$ di borsa. Et per non avere rocti moltiplicha ogni parte per 1100. Averai che 4897 quantità sono iguali a 2596 borsa. Onde quando la borsa vale 4897, la quantità vale 2596. E per lo primo che à 4 quantità $^{823}/_{1100}$ meno una borsa $^{19}/_{25}$, arà 3717. E'l secondo che ponemo aveva una quantità, arà 2596. E el terço, che trovamo aveva $^{13}/_{20}$ quantità $^{4}/_{5}$ borsa arà 5605. E el quarto, che trovamo aveva $^{273}/_{1100}$ di quantità e $^{9}/_{25}$ di borsa aveva 4366. E così e facto. El primo à 3717. E'l secondo à 2596. E el terço à 5605. E el quarto 4366. E la borsa aveva 4897. Che anchora in mi(fol. 271v)nori numeri schifato per 59, el primo à 63, el secondo 44, el terço huomo 95, el quarto huomo 74. E la borsa 83.

(fol. 277r) Quatro huomini ànno denari. E vogliono chomperare uno chavallo. E niuno à tanti denari che'llo possi chonperare. Dice el primo al secondo e al terço huomo, se voi mi date el $^{1}/_{2}$ de' vostri denari, cho'mio io chonperrò el chavallo. Dice el secondo al terço e al quarto huomo, se voi mi date il $^{1}/_{3}$ de' vostri denari io chonperrò el chavallo. Dice il terço huomo al quarto e al primo, se voi mi date $^{1}/_{4}$ de' vostri denari io chonperrò il chavallo. Adima⟨n⟩da anchora il quarto huomo al primo e secondo il $^{1}/_{5}$ de loro denari e dice di chonperare il chavallo. Adimandasi quanti denari aveva ciaschuno e che valeva il chavallo. Benché molti modi sieno ad asolvere tali chasi piglierò el più agevole o vogliamo dire el meno tedioso. Cioè che dirai, noi propogniamo che il primo cho'lla metà delgli denari del secondo cho'l terço à uno cavallo. E diciamo ch'el secondo cho'l terço de' denari del terço e quarto huomo chomper~~ano~~ el chavallo. Adunque tanto à il primo cho'e $^1/_2$ de' denari del secondo e terço huomo quanto à il secondo chon $^1/_3$ del terço e quarto huomo. Onde verai raguagliando, cioè traendo prima da ogni parte $^1/_2$ de' denari del secondo e arremo che' denari del primo chon $^1/_2$ de' denari del terço huomo sono quanto $^1/_2$ de' denari del secondo chon $^1/_3$ del terço e quarto huomo. E dipoi leverai da ogni parte $^1/_3$ de' denari del terço huomo e arremo il primo huomo chon $^1/_6$ de' denari del terço huomo quanto $^1/_2$ de' denari del secondo chon $^1/_3$ de' denari del quarto. E questo nota. E dipoi dirai, noi abbiamo detto che il secondo huomo chon $^1/_3$ de' denari del terço e quarto huomo à quanto il terço huomo chon $^1/_4$ de' denari del quarto e primo huomo, dove tanto sono e denari del secondo huomo chon $^1/_3$ de' denari del terço e quarto huomo quanto sono e denari del terço huomo chon $^1/_4$ de' denari del quarto e primo hu^(fol. 277^v^)omo. Dove raguagliando le parti, da ogni parte leverai $^1/_3$ de' denari del terço huomo e arai e denari del secondo cho'l $^1/_3$ de' denari del quarto essere quanto e $^2/_3$ de' denari del terço uomo chon $^1/_4$ de' denari del quarto e primo huomo. E dipoi da ogni parte leverai $^1/_4$ del quarto huomo, e arai che' denari del secondo huomo chon $^1/_{12}$ del quarto huomo sono quanto $^2/_3$ del terço huomo chon $^1/_4$ de' denari del primo huomo. E questo anchora noterai. E chosì puoi raguagliare l'altre positioni, ma chon queste 2 porrai asolvere. E volendo l'altre aguagliationi fare puoi, che chome vedi nella faccia passata v'è l'aguagliatione del terço. Ora a nostra materia. Noi abbia fatto che' denari del primo chon $^1/_6$ de' denari del terço huomo sono $^1/_2$ de' denari del secondo e $^1/_3$ de' denari del quarto huomo. E anchora abbiamo che e denari del secondo chon $^1/_{12}$ de' denari del quarto huomo sono quanto e $^2/_3$ de' denari del terço huomo e $^1/_4$ de' denari del primo. Onde $^1/_4$ de' denari del primo è d'arre~~g~~chargli a parte de' denari degli altri. Terrai questo modo, noi abbiamo che' denari del primo chon $^1/_6$ de' denari del terço huomo sono quanto il $^1/_2$ de' denari del secondo e $^1/_3$ de' denari del quarto huomo, dove da ogni parte leverai $^1/_6$ de' denari del terço huomo, e arai e denari del primo essere $^1/_2$ de' denari del sechondo e $^1/_3$ de' denari del quarto meno $^1/_6$ de' denari del terço huomo. Onde il $^1/_4$ de' denari del primo huomo sono $^1/_8$ de' denari del secondo e $^1/_{12}$ de' denari del quarto e meno $^1/_{24}$ de' denari del terço huomo. E questo agugnerai a $^2/_3$ de' denari del terço huomo, e arai $^5/_8$ de' denari del terço huomo e $^1/_8$ del secondo e $^1/_{12}$ del quarto. Onde da ogni parte leverai $^1/_8$ del secondo huomo e $^1/_{12}$ del quarto huomo, arai che $^7/_8$ de' denari del secondo sono quanto $^5/_8$ de' denari del terço huomo. Ora q~~uanto~~uesto chonosciuto, quando el secondo huomo avesse 5, el terço arrebbe 7. E avuto questo lume, e noi faremo positione ch'el sechondo huomo avesse 5 quantità, seguita ch'el terço huomo arebbe 7 quantità. [...]

(fol. 278ᵛ)Quatro ànno denari, de' quali vorrebbono chonperare uno chavallo, e niuno di loro à tanto denari che'llo possa chonperare. Dichono il primo e il secondo al terço, se'ttu ci dai il $^1/_3$ de tuoi denari, noi chonperremo el chavallo. Dice il secondo e terço huomo al quarto huomo, se'ttu ci dai il $^1/_4$ de' tuoi denari noi chonperemo el chavallo. Dice il terço e quarto huomo al primo, se'ttu ci dai il $^1/_5$ de' tuoi denari, cho' nostri noi chonperremo el chavallo. Dice il quarto e primo huomo al secondo, se'ttu ci dai il $^1/_6$ de' tuoi denari, noi chonperremo il chavallo. Adimandasi quanto aveva ciaschuno e che valeva il chavallo. Per aguagliatione faremo. Dove dirai, il primo e sechondo chon $^1/_3$ de' denari del terço chonprono il chavallo. E il secondo e terço huomo chon $^1/_4$ del quarto huomo chonprono il chavallo. Adunque tanto sono e denari del primo e secondo cho'l $^1/_3$ de' denari del terço huomo quanto sono e denari del secondo e terço huomo cho'l $^1/_4$ de' denari del quarto. Dove raguagliando le parti, levando da ogni parte e denari del secondo e $^1/_3$ de' denari del terço, aremo che' denari del primo sono quanti e $^2/_3$ de' denari del terço huomo e $^1/_4$ de' denari del quarto huomo. E questo segnia. Dipoi dirai, el secondo e terço huomo chon $^1/_4$ de' denari del quarto huomo chonprono il chavallo. E il terço e quarto huomo chon $^1/_5$ de' denari del primo chonprono uno chavallo. Adunque tanto sono e denari del secondo e terço huomo chon $^1/_4$ de' denari del quarto huomo quanto sono e denari del terço e quarto huomo chon $^1/_5$ de' denari del primo. Onde leverai da ogni parte e denari del terço e $^1/_4$ de' denari del quarto huomo, e aremo che' denari del secondo sono $^3/_4$ de' denari del quarto e $^1/_5$ de' denari del primo. E dipoi seguendo dirai, el terço e quarto huomo chon $^1/_5$ de' denari del primo chonpra il chavallo. E il quarto e primo chon $^1/_6$ de' denari del sechondo chonpra el chavallo. Onde seguita tanto essere e denari del terço e quarto huomo chon $^1/_5$ de' denari del primo quanto il quarto e primo chon $^1/_6$ de' denari del sechondo. Dove raguagliando le parti, levando da ogni parte e denari del quarto huomo e $^1/_5$ del primo, aremo che il terço huomo à e $^4/_5$ del primo e $^1/_6$ del secondo. E questo li segnia. E così farai del quarto huomo, dicendo, el quarto e primo chon $^1/_6$ delli denari del secondo chonpra il chavallo. E'l primo e sechondo cho'l $^1/_3$ de' denari del terço huomo chonpra il chavallo. Onde tanto ànno el quarto e'l primo cho'l $^1/_6$ del denaro del secondo quanto il primo et secondo chon $^1/_3$ de' denari del terço huomo. Onde raguagliando le parti, levando da ogni parte e denari del primo e $^1/_6$ de' denari del secondo, aremo che' denari del quarto sono e $^5/_6$ de' denari del secondo e $^1/_3$ de' denari del terço. È questo notato. E tu inchomincerai alla prima aguagliatione, dicendo, e denari del primo sono e $^2/_3$ de' denari del terço huomo e $^1/_4$ del quarto huomo. Onde è da sapere $^1/_4$ de' denari del quarto, che sono degli altri chontiosiachosa che noi abbiamo trovata che' denari del quarto huomo sono e $^5/_6$ del secondo e $^1/_3$ del terço huomo, dove il $^1/_4$ de' denari del quarto huomo sono quanto e $^5/_{24}$ de' denari del secondo e $^1/_{12}$ de denari' del terço. Dove agli $^2/_3$ de' denari del terço huomo agugnerai $^1/_{12}$ de' denari del terço e $^5/_{24}$ del secondo, fanno $^3/_4$ del terço e $^5/_{24}$ de' denari del secondo. E dipoi arrecha e $^3/_4$ del terço ~~e~~ a parte degli altri, dicendo, el terço huomo à e $^4/_5$ del primo e $^1/_6$ del secondo, dove e $^3/_4$ del terço huomo sono e $^3/_5$ del primo e $^3/_{24}$ del secondo, e agugnerai a $^5/_{24}$ del secondo $^3/_5$ del primo e $^3/_{24}$ del secondo, fanno $^3/_5$ del primo e $^1/_3$ del secondo. E aremo fatto che' denari del primo sono quanto e $^3/_5$ de' denari del primo e $^1/_3$ de' denari del secondo. Dove trarrai da ogni parte $^3/_5$ de' denari del primo, arrai che $^2/_5$ de' denari del primo sono $^1/_3$ de' denari del secondo. Cioè che tantanto[36] sono e $^2/_5$ de' denari del primo quanto il $^1/_3$ de' denari del secondo. Adunque, quando el primo avesse 5, el secondo arebbe 6. Ora per trovare gli altri dirai, el terço à quanto e $^4/_5$ del primo e il $^1/_6$ del secondo, onde e $^4/_5$ del primo e $^1/_6$ del secondo sono 5. E tanto arebbe il terço. E il quarto a $^5/_6$ del secondo e $^1/_3$ del terço. Dove e $^5/_6$ del secondo, sono 5, e il $^1/_3$ del terço huomo sono $1^2/_3$, (fol. 279r)che'n tutto fanno $6^2/_3$. E chosì e fatto, il primo a 5, e il secondo 6, e il terço 5, e il quarto $6^2/_3$. Che per non avere rocti moltiplicha tutti per 3. E arai il primo 15, il secondo 18, e il terço huomo 15 e il quarto huomo 20. E per sapere quanto vale il chavallo agugnerai 15 del primo e 18 del secondo, fanno 33, agli[37] agunto il $^1/_3$ de' denari del terço huomo, cioè di 15, fanno 38. E tanto vale il chavallo. E così ài, il primo aveva 15, e'l secondo 18, e il terço aveva 15, e il quarto huomo aveva 20. E il chavallo valeva[38].

Notes :

1 Actually published in 2021, whence "recent".

2 With a minor modification I follow the definition of what constitutes several *algebraic* unknowns suggested by Albrecht Heeffer [2010: 61]:

(1) The reasoning process should involve more than one rhetorical unknown which is named or symbolized consistently throughout the text. One of the unknowns is usually the traditional *cosa*. The other can be named *quantità*, but can also be a name of an abstract entity representing a share or value of the problem.

(2) The named entities should be used as unknowns in the sense that they are operated upon algebraically by arithmetical operators, by squaring or root extraction. [...].

(3) The determination of the value of the unknowns should lead to the solution or partial solution of the problem. [...].

(4) The entities should be used together at some point of the reasoning process and connected by operators or by a substitution step.

For reasons that will be obvious in the following I shall disregard the word "rhetorical" in (1)

3 A thoroughly documented biography of Benedetto can be found in [Ulivi 2002]. He was almost certainly born in 1429 and died in 1479.

Repeating the erroneous information given in his manuscript, Arrighi ascribes the *Tractato d'abbacho* to Pier Maria Calandri. The mistake is corrected in [Van Egmond 1980: 96].

4 My translation, as all translations into English in what follows.

Nesselmann's application of his own division into rhetorical, syncopated and symbolic algebra can be discussed – not least because of the restricted range of sources available to him in 1842. However, most of those who reject the scheme as outdated do so because they have not understood the quotation that follows here.

5 My translations of Benedetto's texts are made so as to agree very closely with the Tuscan originals in order to facilitate comparison even for readers with only rudimentary familiarity with modern Italian or Renaissance Tuscan (any of the two will do, the difference is modest). Not least because Benedetto's grammar is less perfect than his mathematics, the result is often clumsy and grammatically inconsistent; this notwithstanding the translations are probably neither more nor less incomprehensible than Benedetto's words will have been for fifteenth-century readers.

The translation follows Benedetto's corrected text, tacitly omitting what he has deleted (the deletions can be found in the transcription in the appendix). Words in ⟨ ⟩ have been forgotten by Benedetto; they are restored from his preceding symbolic calculations and after further control that they are presupposed in what follows. Words in ^ ^ have been forgotten and then inserted between the lines by Benedetto; passages marked * * were at first forgotten and were then written in the margin. Errors Benedetto has overlooked are conserved but pointed out in notes – they as well as all other corrections all concern the secondary description in words, not the original symbolic calculations. Benedetto has a rudimentary punctuation, but from a modern point of view is it poor and inconsistent; the present punctuation replaces it. For the sake of layout convenience, the horizontal fraction lines of the manuscript have been replaced by slashes.

6 The *denaro* is a specific monetary value (corresponding to the penny in the classical British system); but the plural *denari* (occasionally also the singular) has the further generic meaning "money". This dual interpretation calls for the use of a loanword (*pennies* in the same double function seems to have gone out of use).

7 Benedetto's fractions are often preceded by a definite article. The implication is that, for example, "the $^1/3$ " is supposed to be read *il terço*, "the third", not "the one-third". Other abbacus texts (not Benedetto) also often write (for example, again) $^1/_3$ when intending the ordinal "third", for example "third man".

8 Technically, this can be seen to mean that it is written down as a symbolic equation. For further clarification, see note 10.

9 Technically, as we see in the following, this refers to an algebraic substitution.

10 The verb *notare* may mean "write down" as well as "consider". "Take note" should be similarly ambiguous. *Notare* as well as the functionally similar *segnare* (above, note 8) are used about the writing of simplified equations.

11 *Castelluccio*, "small castle". The word obviously refers to the marginal calculations on the previous page, well enclosed by lines as if composed of several courtyards or chambers. Here, as we see, Benedetto states explicitly that the marginal calculation is already there when the main text is written.

12 In some of Benedetto's preceding problem solutions with two algebraic unknowns, these are the *chosa*, "thing", and *quantità,* "quantity", cf. above. In others, his unknowns are *quantità* and *borsa*. As we see, Benedetto starts from the first routine and then chooses the other, allowing him to conserve the *borsa*.

13 Error for " $^{19}/25$ of purse". The marginal calculation, correctly, gives "δ $^{272}/_{1100}$ *q* $^{19}/_{25}$ *b*".

14 Error for "the third", as confirmed by the following words.

15 As we see, the process of "confronting"/*ragugliamento* refers to the process of constructing the *reduced* equation – in the present case by addition as well as subtraction.

15 Equality is indicated by large distance, addition by close juxtaposition.

17 Heeffer [2010 : 90] locates an instance with four unknowns *A*, *B*, *C* and *D* on p. 194 in Johannes Buteo's "De regula quantitatis", the last section of book III of his *Logistica* from [1559] (it is preceded by examples with two and three unknowns). Even here, there is a symbolic calculation accompanying an explanation in words, which from internal evidence can be seen to have been written after the calculation in symbols (since the book is printed we have no direct way to ascertain which was written first).

Buteo uses the method of two to four unknowns to linear problems only, and thus does not go beyond Benedetto mathematically (his symbolic calculations are very similar to those of Benedetto); nor does Guillaume Gosselin [1577: fols. 79^{v}–86^{v}]. We have to wait another three decades before Viète [1591] made the jump to higher degrees.

One might claim that Stifel goes beyond Benedetto in the *Arithmetica integra* [1544: 251^{v}–252^{r}]. Here, inspired by Christoph Rudolff's and Girolamo Cardano's naming of two unknowns (*thing* and *quantity*) he introduces a way to give names to as many algebraic unknowns as wanted, and even to their powers and products. However, apart from a reducible quartic problem on fol. 254^{v}, which is solved by means of geometry and not by algebra, the examples which Stifel uses as illustrations are of the first degree, and quite trivial in comparison to what is offered by Benedetto (and later by Buteo). It is for good reasons, we may say, that Moritz Cantor [1892: 405] dedicates only 5 lines to Stifel's naming (not discussing his use), Kurt Vogel [1976: 60] no more, and that nobody else whom I know about has taken notice at all in writing; Joseph E. Hofmann [1868: 32*f*] dedicates 7 lines to the subject.

Much earlier than Benedetto, very well known in Antiquity as well as the Latin High Middle Ages but never inspiring any emulation, is a rudimentary algebraic argument in Aristotle's *Nicomachean Ethics* 1133^{b}22– 26 [trans. Barnes 1984: 75] making use of single-letter representations:

> Let A be a house, B ten minae, C a bed. A is half of B, if the house is worth five minae or equal to them; the bed, C, is a tenth of B; it is plain, then, how many beds are equal to a house, viz. five.

The idea was there, ready to be picked up and generalized by anybody who was interested. Nobody was.

[18] Evidently, the same objection could be raised to Benedetto's verbal description of the procedure in the problem we have analyzed. However, since this description is secondary it changes nothing in the characterization of the marginal calculation as a perfect symbolic algebraic calculation with five unknowns.

[19] Florence, Biblioteca Medicea Laurenziana, Gaddi 36.

[20] See the explanation in [Høyrup 2019: 153].

[21] A closely related problem is found in the *Liber abbaci* [ed. Boncompagni 1857: 240; Giusti 2020: 393]; there, the fractions are $^1/_3$, $^1/_4$, $^1/_5$ and $^1/_6$ – apart from that the structure of the question is identical. There too, Fibonacci's procedure is if not algebraic than at least quasi-algebraic, but even in the present case Benedetto calculates on his own.

[22] Addition is still indicated by juxtaposition, but now equality first by "*iguali* ——", then by —— alone. At the time, this long stroke was beginning to be regularly used to indicate equality in symbolic calculations (but occasionally also for other confrontations). Though not used in the initial equations of the purse problem discussed above, it also turns in the later part of the calculations (those following upon the introduction of the unknown *quantity*) – as well as elsewhere in Benedetto's manuscript, before and after the present location.

[23] Or "make (reduced) equations of" – I have been unable to find an English translation reflecting both senses.

[24] *Aguagliazioni* – that is, the (reduced) equations resulting from the process of *ragugliamento*, "confrontation".

[25] Namely in the three lines of the marginal calculation that follow immediately after the two times three lines rendered above – that is, the last three lines of the redrawn manuscript excerpt. We may presume that Benedetto after having made all three sets found out in the ensuing calculations that he did not need the last of them. We may remember his words "It is true that it could be done without that, but since it is in the castelet, this order shall be pursued" in his solution to the problem discussed previously.

[26] We may observe that expressions of the type "*a* and less *b*" imply thinking in terms of subtractive (not necessarily negative) numbers; this is not sensational and no idiosyncrasy on the part of Benedetto.

[27] *per aguagliatione* – as we remember and as it is confirmed by the ensuing text, this term refers to the *reduced* equation (as a rule at least).

[28] On *segnare*/"to mark", cf. note 10. From my scan it seems that the underlining of the equations that are "marked" is heavier that the lines serving the subtraction – see the redrawn calculation. However, this possibly emphatic underlining (which may also be a mere separation of subsections of the calculation) is hardly what the verb refers to directly; the parallelism suggests it to be in the present context a mere synonym for *notare*/"to take note of".

[29] *aguagliatione* – as we see, actually the first *reduced* equation.

[30] There are a few inconsequential exception, on fols 314^{r}, 315^{r}, 325^{r} and 335^{r}. In all three cases, composite geometric diagrams (no calculations) seem to have been made first – most likely because it was difficult to predict how much space a diagram would take up before it was effectively drawn.

[31] The slate was in use for arithmetical computation in Italy at least since 1410, see [Smith 1923: II, 179].

[32] For simplicity I show an example with three unknowns [Buteo 1559: 190*f*]. Three numbers are to be found such that the first with 1/3of the others makes 14, the second with 1/4of the others makes 8, and the third with 1/5of the others makes 8. This is reformulated in three equations (Buteo indicates addition by "." or ",", while "[" is his equation sign):

$$\begin{array}{l} 3A.\ 1B.\ 1C\ [\ 42 \\ 1A.\ 4B.\ 1C\ [\ 32 \\ 1A.\ 1B.\ 5C\ [\ 40 \end{array}$$

Next follows

$$\begin{array}{r} 3A,\ 12B,\ 3C\ [\ 96 \\ \underline{3A,\ \ 1B,\ 1C\ [\ 42} \\ 11B,\ 2C\ [\ 54 \end{array}$$

$$\begin{array}{r} 3A.\ 3B.\ 15C\ [\ 120 \\ \underline{3A.\ 1B.\ \ 1C\ [\ 42} \\ 2B.\ 14C\ [\ 78 \end{array}$$

$$\begin{array}{r} 22B.\ 154C\ [\ 858 \\ \underline{22B.\ \ \ 4C\ [\ 108} \\ 150C\ [\ 750 \end{array}$$

Buteo now concludes that C is 750÷150 = 5, and then finds B from $2B.\ 14C\ [\ 78$, etc.

[33] The problem is once again borrowed from the *Liber abbaci* [ed. Boncompagni 1857: 243; ed. Giusti 2020: 398]. Fibonacci solves it by a quasi-algebraic procedure.

[34] Error for " $^{19}/_{25}$ *di borsa*". The marginal calculation, correctly, gives "δ $^{272}/1100$ *q* $^{19}/_{25}$ *b*".

[35] Error for "il terço", as confirmed by the following words.

[36] Error for *tanto*.

[37] Or, abbreviated, "a quelli". The meaning does not change but the grammar would be better.

References

[1] Arrighi, Gino (ed.), 1974. Pier Maria Calandri, *Tractato d'abbacho*. Dal codice Acq. e doni 154 (sec. XV) della Biblioteca Medicea Laurenziana di Firenze. Pisa: Domus Galilaeana.

[2] Barnes, Jonathan (ed.), 1984. The Complete *Works* of Aristotle. The Revised Oxford Translation. 2 vols. Princeton: Princeton University Press.

[3] Boncompagni, Baldassare (ed.), 1857. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. I. Il *Liber abbaci* di Leonardo Pisano. Roma: Tipografia delle Scienze Matematiche e Fisiche.

[4] Buteo, Joannes, 1559. *Logistica, quae et arithmética vulgò dicitur in libros quinque digesta*. Lyon: Guillaume Rouillé.

[5] Cantor, Moritz, 1892. *Vorlesungen über Geschichte der Mathematik*. Zweiter Band. *Von 1200–1668*. Leipzig: Teubner.

[6] Giusti, Enrico (ed.), 2017. "The Twelfth Chapter of Fibonacci's *Liber abbaci* in Its 1202 Version". *Bollettino di Storia delle Scienze Matematiche* **37**, 9–216.

[7] Giusti, Enrico (ed.), 2020. Leonardi Bigolli Pisano *vulgo* Fibonacci *Liber Abbaci*. Firenze: Leo S. Olschki.

[8] Gosselin, Guillaume, 1577. *De arte magna, seu de occulta parte numerorum quae et Almucabala vulgo dicitur*. Paris: Egide Beys, 1577.

[9] Heeffer, Albrecht, 2010. "From the Second Unknown to the Symbolic Equation", 57–101 *in* Albrecht Heeffer & Maarten Van Dyck (eds), *Philosophical Aspects of Symbolic Reasoning in Early Modern Mathematics*. London: College Publications.

[10] Hofmann, Joseph Ehrenfried, 1968. *Michael Stifel, 1487?-1567: Leben, Wirken und Bedeutung für die Mathematik seiner Zeit*. Wiesbaden: F. Steiner.

[11] Høyrup, Jens, 2019. "Reinventing or Borrowing Hot Water? Early Latin and Tuscan Algebraic Operations with Two Unknowns". *Gaṇita Bhāratī* **41**, 115-159.

[12] Nesselmann, G. H. F., 1842. *Versuch einer kritischen Geschichte der Algebra*. Nach den Quellen bearbeitet. Erster Theil, *Die Algebra der Griechen*. Berlin: G. Reimer.

[13] Smith, David Eugene, 1923. *History of Mathematics*. I. *General Survey of the History of*

Elementary Mathematics. II. *Special Topics of Elementary Mathematics*. Boston: Ginn, 1923–1925.

[14] Ulivi, Elisabetta, 2002. "Benedetto da Firenze (1429–1479), un maestro d'abaco del XV secolo. Con documenti inediti e con un'Appendice su abacisti e scuole d'abaco a Firenze nei secoli XIII–XVI". *Bollettino di Storia delle Scienze Matematiche* **22**:1, 3–243.

[15] Van Egmond, Warren, 1980. *Practical Mathematics in the Italian Renaissance: A Catalog of Italian Abbacus Manuscripts and Printed Books to 1600*. Firenze: Istituto e Museo di Storia della Scienza.

[16] Viète, Franḍois, 1591. *In artem analyticem isagoge*. Tours: Jamet Mettayer.

[17] Vogel, Kurt, 1976. "Stifel", pp. 58–62 *in Dictionary of Scientific Biography*, vol. XIII. New York: Scribner.

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